

# Acceleration in a Rotating Frame

We have seen already that the acceleration of a particle of fluid whose velocity  $\vec{v} = \vec{v}(t, x, y, z)$  is

$$\vec{a} = \frac{d\vec{v}}{dt} + \vec{v} \cdot \nabla \vec{v}.$$

In component form,  $\vec{a}$  is:

$$\left\{ \begin{array}{l} a_1 = \frac{dv_1}{dt} + v_1 \frac{dv_1}{dx} + v_2 \frac{dv_1}{dy} + v_3 \frac{dv_1}{dz} \\ a_2 = \frac{dv_2}{dt} + v_1 \frac{dv_2}{dx} + v_2 \frac{dv_2}{dy} + v_3 \frac{dv_2}{dz} \\ a_3 = \frac{dv_3}{dt} + v_1 \frac{dv_3}{dx} + v_2 \frac{dv_3}{dy} + v_3 \frac{dv_3}{dz}. \end{array} \right. \quad (1)$$

Here we want to find the modifications we have to make to (1) in order to allow the frame on which the particle is located to rotate with angular velocity  $\vec{\Omega}$ .

First picture our planet rotating with its usual angular velocity ( $\vec{\Omega} = \vec{\Omega}_E$ ,  $\Omega = 0.000727 \text{ rad/sec}$ ) and let  $P$  be a particle that is stationary with

respect to the planet.

Let  $x, y, z$  be an inertial coordinate frame. Let  $P$  have co-latitude  $\phi$  at. The relative to the inertial coordinate axes  $P$  traces a curve  $C$  ~~at time shown here.~~<sup>→</sup> The parametrization of  $C$  can be parametrized as

$$\vec{r}(t) = \langle R \cos \Omega t \sin \phi, R \sin \Omega t \sin \phi, R \cos \phi \rangle$$

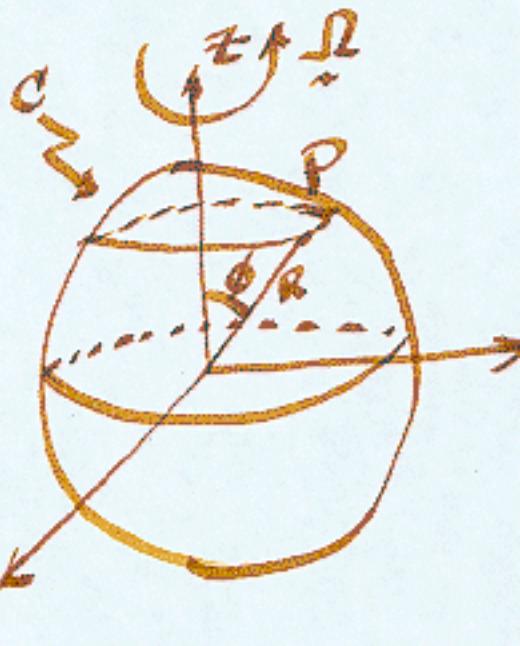
where  $R$  is the radius of the Earth and  $\Omega t$  is the measure of  $P$ 's longitude at time  $t$ . Because the position of  $P$  is changing with time,  $P$  has a velocity in the inertial frame given by

$$\frac{d\vec{r}}{dt} = \langle -\Omega R \sin \Omega t \sin \phi, \Omega R \cos \Omega t \sin \phi, 0 \rangle. \quad (2)$$

It is easy to show that (2) is equivalent to

$$\frac{d\vec{\Sigma}}{dt} = \vec{\Omega} \times \vec{\Sigma}. \quad (3)$$

So the rotation of the Earth induces a velocity  $\vec{\Omega} \times \vec{\Sigma}$  in



(3) way particle that is located in position  $\underline{r}$ .

Differentiating (2) again yields:

$$\begin{aligned}\underline{a} = \frac{d^2\underline{r}}{dt^2} &= -\Omega^2 \langle R \cos \Omega t \sin \phi, R \sin \Omega t \sin \phi, 0 \rangle \\ &= -\Omega^2 \underline{R},\end{aligned}$$

where  $\underline{R}$  is the projection of  $\underline{r}$  on the  $xy$ -plane, i.e.,

$$\underline{R} = \langle R \cos \Omega t \sin \phi, R \sin \Omega t \sin \phi, 0 \rangle.$$

It is also easy to show that

$$\frac{d^2\underline{r}}{dt^2} = \underline{\Omega} \times (\underline{\Omega} \times \underline{r}).$$

So  $\underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = -\Omega^2 \underline{R}$  is the acceleration imparted on a stationary particle by the rotation of the Earth. This vector has its largest value at the equator (where  $\underline{r}$  has its largest projection) and that value can be estimated as

$$\|-\Omega^2 \underline{R}\| = (6,000,000)^2 (6,000,000) \approx 0.03 \text{ m/s}^2$$

for a 6,000 Kilometer planet, radius. Relative to  $g = 9.8 \text{ m/s}^2$ ,  $-\Omega^2 \underline{R}$  is rather small and is often neglected.

the expression  $\Omega \times (\Omega \times \vec{r})$  is often called  
the centrifugal acceleration.

Now let's calculate the acceleration of a particle of fluid that moves relative to the Earth. Let  $\vec{v}$  be the velocity of the particle relative to the rotating frame (this is in fact the velocity we compute when we lower a CTD into a current). Taking (3) into account, the absolute velocity of the particle, as measured in the inertial frame, is

$$\vec{v}_a = \vec{v} + \frac{d}{dt} \times \vec{r}. \quad (5).$$

It helps to understand that the relative velocity  $\vec{v}$  is nothing but  $\frac{dr}{dt}$ , where now  $r(t)$  is the parametrization of the particle's motion relative to the Earth. Define the equation

$$\frac{D}{Dt} = \frac{d}{dt} + \frac{\Omega}{2} \times$$

$\frac{D}{Dt}$  is the absolute time differentiation, while  $\frac{d}{dt}$  is the relative (to Earth) time differentiation.

To determine the absolute acceleration of a

particle we must compute  $\frac{D^2}{Dt^2}$  of  $\vec{v}(t)$ .

$$\frac{D^2 \vec{r}}{Dt^2} = \left( \frac{d}{dt} + \vec{\Omega} \times \right) \left( \frac{d}{dt} + \vec{\Omega} \times \right) \vec{r}$$

$$= \left( \frac{d}{dt} + \vec{\Omega} \times \right) \left( \vec{v} + \vec{\Omega} \times \vec{r} \right)$$

$$= \frac{d\vec{v}}{dt} + \vec{\Omega} \times \vec{v} + \vec{\Omega} \times \vec{v} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}),$$

where I have used  $\frac{d\vec{v}}{dt} = \vec{v}$  and  $\frac{d\vec{\Omega}}{dt} = 0$  (a good assumption on the Earth!). So

$$\frac{D}{Dt} \vec{a} = \frac{d\vec{v}}{dt} + 2 \vec{\Omega} \times \vec{v} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}).$$

The first term is  $\vec{a}$ ,  $\frac{d\vec{v}}{dt}$ , is our old acceleration as defined in (1). The third term,  $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ , is the centripetal acceleration, which we agreed to neglect. The second term,  $2 \vec{\Omega} \times \vec{v}$ , is called the Coriolis Force, which is arguably one of the most important terms we need to consider when modeling ocean currents. So, in our equations of motion we will use

$$\frac{d\vec{v}}{dt} + 2 \vec{\Omega} \times \vec{v}$$

for acceleration.